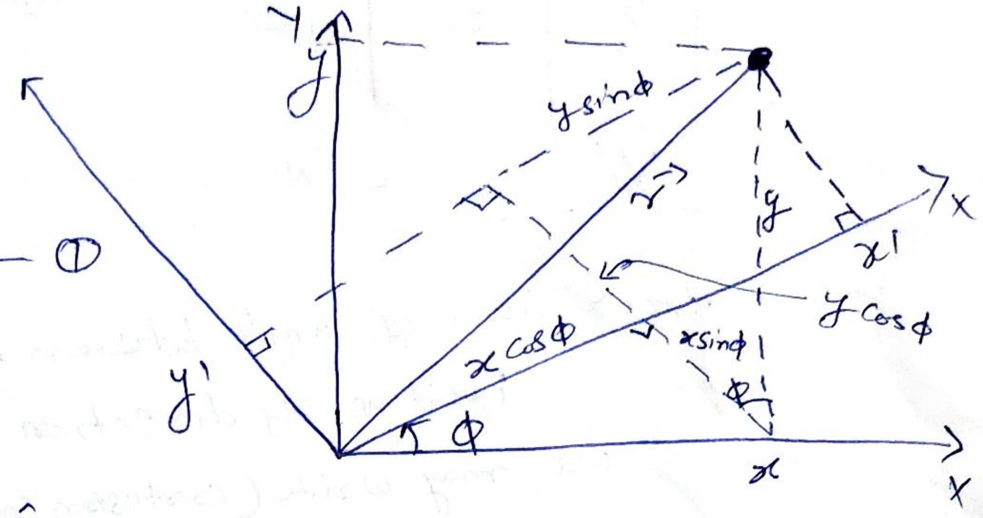
① Rotation of Coordinate axes in 2D ( )

$\vec{r} = x\hat{i} + y\hat{j}$

$$\left.\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi \end{aligned}\right] — ①$$



Take vector $\vec{A}$

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

In rotated coordinate system $\vec{A'} = A_x' i + A_y' j$

$$\left.\begin{aligned} A_x' &= A_x\cos\phi + A_y \sin\phi \\ A_y' &= -A_x \sin\phi + A_y \cos\phi \end{aligned}\right] — ②$$

From ② $\quad A_x'^2 + A_y'^2 = A_x^2 + A_y^2$

$\Rightarrow |\vec{A}| = |\vec{A'}|$ { Scalar is invariant

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} — ③$$ { To get $\begin{bmatrix} x \\ y \end{bmatrix}$ find inverse of matrix

Now $\quad x \rightarrow x_1 \quad y \rightarrow x_2$ $\quad$ | $a_{11} = \cos\phi \quad a_{12} = \sin\phi$, $\quad a_{21} = -\sin\phi, \quad a_{22} = \cos\phi$

we can write ③ as

$$x_i' = \sum_{j=1}^{2} a_{ij} x_j \;; \quad i = 1, 2 \; — ④$$

$a_{ij} \rightarrow$ direction cosine.

$$a_{ij} \equiv \cos(x_i', x_j)$$

$$a_{11} = \cos(x_1', x_1) = \cos\phi$$

$$a_{12} = \cos(x_1', x_2) = \cos(\phi - \pi/2) = \sin\phi$$

$$\begin{aligned} a_{21} &= \cos(x_2', x_1) \\ &= \cos(y', x) \\ &= \cos(\tfrac{\pi}{2} + \phi) \\ &= -\sin\phi \end{aligned}$$

Now generalized ④ to N dim

$$V_i' = \sum_{j=1}^{N} a_{ij} V_j \quad , i = 1, 2, \ldots, N \quad \text{—— (5)}$$

$$\begin{bmatrix} V_1' \\ V_2' \\ \vdots \\ V_N' \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & a_{2N} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}_{N \times N} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$

a_{ij} → Cosine of angle between the x_i' direction and positive x_j direction.

We may write (cartesian coordinates)

$$a_{ij} = \frac{\partial x_i'}{\partial x_j'}$$

using inverse rotation [$\phi \rightarrow -\phi$]

$$x_j = \sum_{i=1}^{2} a_{ij} x_i' \Rightarrow \frac{\partial x_j'}{\partial x_i'} = a_{ij}$$

$$\text{Now } V_i' = \sum_{j=1}^{N} \frac{\partial x_i'}{\partial x_j} V_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x_i'} V_j \quad \text{—— (6)}$$

The direction cosine a_{ij} satisfy (orthogonality condition)

$$\left.\begin{array}{l} \sum_i a_{ij} a_{ik} = \delta_{jk} \\ \text{or } \sum_i a_{ji} a_{ki} = \delta_{jk} \end{array}\right] \text{—— (7)}$$

$\delta_{jj} = 1$, for $j = k$

$\delta_{jk} = 0$, for $j \neq k$

Gradient

Total variation of a scalar point function $\phi(x,y,z)$

$$d\phi(x,y,z) = [\phi(x+dx, y+dy, z+dz) - \phi(x, y+dy, z+dz)]$$
$$+ [\phi(x, y+dy, z+dz) - \phi(x, y, z+dz)]$$
$$+ [\phi(x, y, z+dz) - \phi(x, y, z)]$$

$$d\phi(x,y,z) = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$\because$ $dF(x,y) = F(x+dx, y+dy) - F(x,y)$

$$= [F(x+dx, y+dy) - F(x, y+dy)] + [F(x, y+dy) - F(x,y)]$$

$\uparrow$ adding and substracting this

$$= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Since ϕ is scalar

$$\phi'(x_1', x_2', x_3') = \phi(x_1, x_2, x_3)$$

$$\frac{\partial \phi'(x_1', x_2', x_3')}{\partial x_i'} = \frac{\partial \phi(x_1, x_2, x_3)}{\partial x_i'} = \sum_j \frac{\partial \phi}{\partial x_j'}\frac{\partial x_j'}{\partial x_i'} = \sum_j a_{ij}' \frac{\partial \phi}{\partial x_j'}$$

by eq^n (6) we have constructed a vector with component $\frac{\partial \phi}{\partial x_j}$. This vector we label the gradient of ϕ

$$\nabla \phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

$$\nabla = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

$\uparrow$ vector differential operator

Importance
$\vec{F} = -\vec{\nabla} U$